

Some thick brane solutions in $f(R)$ -gravity

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The thick brane model is considered in $f(R) \sim R^n$ gravity. It is shown that regular asymptotically anti-de Sitter solutions exist in some range of values of the parameter n . A peculiar feature of this model is the existence of a fixed point in the phase plane where all solutions start, and the brane can be placed at this point. The presence of the fixed point allows to avoid fine tuning of the model parameters to obtain thick brane solutions.

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I. INTRODUCTION

At the present time, the study of the structure and evolution of the Universe is at an interesting stage. The necessity of a consistent description of the present Universe demands the creation of a unified theory of elementary particles and cosmology. A very promising way is the consideration of models of the Universe in higher-dimensional theories. Such investigations were initiated in the works of Kaluza and Klein in the 1920s for the unification of the two fundamental interactions known at that time - gravitation and electromagnetism - within the framework of a unified five-dimensional theory. Later on, similar ideas were used for the unified description of the four currently known fundamental interactions within the framework of superstring theories with several extra space dimensions. As in the case of the Kaluza-Klein theories, in superstring theories, it is supposed that our four-dimensional space-time results from the spontaneous compactification of a higher-dimensional space.

At the same time, models of the Universe with non-compact (and even infinite) extra dimensions are under consideration [1] (for a review, see also [2, 3]). In such a theory, it is supposed that we live on a thin leaf (brane) embedded into some higher-dimensional space (bulk), and matter is somehow confined (trapped) on the brane. The existence of extra dimensions then allows to resolve a number of old problems in high-energy physics (such as the problem of mass hierarchy, stability of the proton, etc.).

All branes can be divided into two classes: thin branes and thick branes. In the first case, one has a delta-like localization of matter on the brane [1]. From a realistic point of view, however, the brane should have some thickness. The inclusion of a brane thickness then yields new possibilities and new problems (for a review, see, e.g., [4]). Such a brane must satisfy two major requirements: 1) the solutions should be regular and asymptotically flat, or de Sitter ones (or anti-de Sitter ones); 2) ordinary matter should be confined to the brane.

Most thick brane models employ scalar fields within the framework of Einstein's theory of gravity (see e.g. the review [4] and references therein). However, one might expect the existence of brane-like solutions also for some kinds of modified gravity theories, the so-called higher-order gravity theories (HOGT). In such theories the action of the Einstein-Hilbert gravitational Lagrangian is supplemented by further terms, which are curvature invariants [5]. (Such a modification is based on the effect of the interaction of quantum matter fields with the classical gravitational field.) This allowed to avoid an initial cosmological singularity and to construct regular cosmological models of the early Universe [6]. Later it was shown that in such type of models a stage of inflation can exist [7].

Currently, this last possibility is widely used for the description of the present accelerated expansion of the Universe. This acceleration can be explained by the presence of some antigravitating substance - the so-called dark energy. The description of dark energy can also be realized within the framework of $f(R)$ theory, where $f(R)$ is some arbitrary function of the scalar curvature R . By choosing $f(R) \sim R^n$, it was shown that such models are in good agreement with several different sets of observations [8]-[12]. On the other hand, such theories can be successfully employed for the description of dark matter [13] as well.

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HOGT with more complicated combinations of curvature invariants are also under consideration. In particular, in the low-energy limit of M-theory the Gauss-Bonnet invariant appears

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

It was shown that such models, on the one hand, do not contradict observations within the Solar System and, on the other hand, successfully describe the present accelerated expansion of the Universe [14]. These models can be used in the description of an effective equation of state both for an effective cosmological constant and for the dynamic case (quintessence, phantom dark energy), and also for the description of the transition of one type of dark energy (quintessence) into another one (phantom energy). Also there are theories which employ both $f(R)$ and Gauss-Bonnet terms to describe dark energy [15].

Another usage of HOGT consists in the consideration of higher-dimensional cosmological and astrophysical models. In particular, in Ref. [16] brane world and black hole models with higher-dimensional action

$$S = \int d^d x \sqrt{-g} \left[aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{k^2}R - \Lambda + L_m \right],$$

(where L_m is the matter Lagrangian; a, b, c are arbitrary constants) were considered. The obtained results allow to estimate general properties of models within the framework of HOGT.

Furthermore there are some results obtained by the application of HOGT for the creation of brane-world models [17]. In particular, Parry et al. [18] considered a brane model in $f(R) = R + \alpha R^2$ theory. Making use of the conformal equivalence of such gravity models and Einstein-Hilbert gravity with a scalar field, the authors rewrote the $f(R)$ -equations in the form of the Einstein equations with some scalar field source. They showed that in such models brane-like solutions exist.

Whereas Parry et al. [18] considered thin branes, in this paper we investigate thick branes in such $f(R)$ theories to see, whether this leads to new and physically more acceptable results.

II. EQUATIONS AND SOLUTIONS IN $f(R) \sim R^n$ THEORY

We will work in a five-dimensional spacetime. The corresponding gravitational action can be taken in the form

$$S = \int d^5 x \sqrt{-^5g} \left[-\frac{R}{2} + f(R) \right], \quad (1)$$

where $f(R)$ is an arbitrary function of the scalar curvature R . (Here we employ units such that $8\pi G = c = 1$.) Variation of the action (1) with respect to the 5-dimensional metric tensor g_{AB} led to gravitational equations:

$$R_A^B - \frac{1}{2}\delta_A^B R = \hat{T}_A^B, \quad (2)$$

where capital Latin indices run over $A, B, \dots = 0, 1, 2, 3, 5$, and

$$\hat{T}_A^B = - \left\{ \left(\frac{\partial f}{\partial R} \right) R_A^B - \frac{1}{2}\delta_A^B f + (\delta_A^B g^{LM} - \delta_A^L g^{BM}) \left(\frac{\partial f}{\partial R} \right)_{;L;M} \right\} \quad (3)$$

specifies the effective geometric matter source with the nontrivial dependence on curvature; the semicolon denotes the covariant derivative. One can see that the gravitational equations in $f(R)$ -gravity rewritten in the form (2) have a structure that coincides with the standard general relativity equations when the source of the gravitational field is the effective energy-momentum tensor (3). One can check that the energy-momentum conservation law is satisfied as well (see, e.g., [6]).

Here we will focus on a special choice of $f(R)$ in the form

$$f(R) = -\alpha R^n, \quad (4)$$

where $\alpha > 0$ and n are constants. As shown in Refs. [8], where the present accelerated expansion of the Universe was considered, there are some ranges of n which do not contradict the observational cosmological data. Therefore it seems natural to consider these values of n for brane models as well.

We will adopt the flat brane model with the metric

$$ds^2 = e^{2y(z)} \eta_{\alpha\beta} dx^\alpha dx^\beta - dz^2, \quad (5)$$

where the warp factor function depends on the fifth coordinate z only, and $\eta_{\alpha\beta} = \{1, -1, -1, -1\}$ is a Minkowski metric. Inserting this metric into Eqs. (2) and (3), one obtains from the (z) component of the Einstein equations

$$p \frac{d^2 p}{dy^2} + \left(\frac{dp}{dy} \right)^2 + 5p \frac{dp}{dy} = \frac{1}{32p^2 f_{RR}} \left[4p \left(\frac{dp}{dy} + p \right) f_R - \frac{1}{2} f - 6p^2 \right]. \quad (6)$$

where the new variable $p = dy/dz$ was introduced, and the index R denotes the derivative with respect to the scalar curvature R . Eq. (6) is a third-order differential equation with respect to the metric function y , and the remaining components of the Einstein equations are fourth-order ones. Using the expression for $f(R)$ from (4) one can obtain the equation for y in the form

$$y''' - \frac{1}{n} \frac{y''^2}{y'} + \left[5 - \frac{\frac{7n}{2} - 5}{n(n-1)} \right] y' y'' - \frac{5}{2} \frac{n - \frac{5}{2}}{n(n-1)} y'^3 = \frac{12y'}{\alpha 8^n n(n-1)} \left(y'' + \frac{5}{2} y'^2 \right)^{2-n}, \quad (7)$$

where the prime denotes the derivative with respect to z . We note, that by introducing the scaled variable $\bar{z} = \bar{\alpha} z$ with $\bar{\alpha} = \alpha^{\frac{1}{2(1-n)}}$, Eq. (7) becomes independent of α . Thus it is sufficient to solve the equation for $\alpha = 1$. All other solutions are obtained from this solution by scaling.

One can also see from Eq. (7) that the first derivative y' cannot take the value zero unless $y''(z) = 0$ as well. As will be shown below there is only one point in the phase plane where both y' and y'' are equal to zero - the fixed point.

Due to its complexity, we will seek numerical solutions of Eq. (7). But before, let us investigate the qualitative behavior of solutions of Eq. (7). For this purpose, we rewrite it as a system of three first-order differential equations

$$\begin{aligned} y' &= p, \\ p' &= v, \\ v' &= \frac{1}{n} \frac{v^2}{p} - \left[5 - \frac{\frac{7n}{2} - 5}{n(n-1)} \right] p v + \frac{5}{2} \frac{n - \frac{5}{2}}{n(n-1)} p^3 + \frac{12p}{\alpha 8^n n(n-1)} \left(v + \frac{5}{2} p^2 \right)^{2-n}. \end{aligned} \quad (8)$$

The fixed point of the system is

$$\mathcal{A} = \{p \rightarrow 0, v \rightarrow 0\}. \quad (9)$$

In order to analyze the behavior of solutions at the fixed point, let us consider a variation $\delta \ll y$ in the neighborhood of the fixed point for Eq. (7). The corresponding equation for δ then becomes

$$\delta''' - \frac{2}{n} \frac{y''}{y'} \delta'' + \frac{1}{n} \frac{y''^3}{y'^2} \delta' = 0. \quad (10)$$

Taking into account that at the fixed point denoted by $z = z_{fp}$ the derivatives vanish, i.e., $y'_{fp} = 0$ and $y''_{fp} = 0$, we seek a solution of Eq. (7) in the neighborhood of the fixed point of the form

$$y = y_{fp} + y'''_{fp} \frac{(z - z_{fp})^3}{6}.$$

Then Eq. (10) takes the form

$$\delta''' - \frac{4}{n(z - z_{fp})} \delta'' + \frac{4y'''_{fp}}{n(z - z_{fp})} \delta' = 0$$

with the solution

$$\delta = \delta_0 (z - z_{fp})^{2+4/n}.$$

To be sure that this solution decays faster than y , it is necessary to suppose that

$$2 + \frac{4}{n} > 3 \quad \Rightarrow \quad 0 < n < 4, \quad n \neq 1. \quad (11)$$

Let us next analyze the behavior of solutions in the plane $\{v, p\}$ near the fixed point following the approach of [19]. As an example, let us consider the important case with $n = 2$ (and $\alpha = 1$). In this case Eq. (7) takes the form

$$p'' + f(p, p')p' + g(p) = 0,$$

or, equivalently,

$$p' = v, \quad v' = -g(p) - f(p, v)v,$$

where

$$f(p, p') = -\frac{1}{2} \frac{p'}{p} + 4p, \quad g(p) = \frac{5}{8}p^3 - \frac{3}{32}p.$$

It is obvious from the above expression that

$$\int_0^{\pm\infty} g(p)dp = G(\pm\infty) = +\infty.$$

This is a necessary condition to be satisfied in the analysis of such systems [19]. According to [19], in this case the origin of the coordinates is then a single fixed point. It is surrounded by the so-called “curves of energy” with the equation

$$w(p, v) \equiv G(p) + \frac{1}{2}v^2 = C.$$

All of these are closed curves. In our case the function $g(p)$ is odd, and therefore these curves are symmetrical with respect to both coordinate axes.

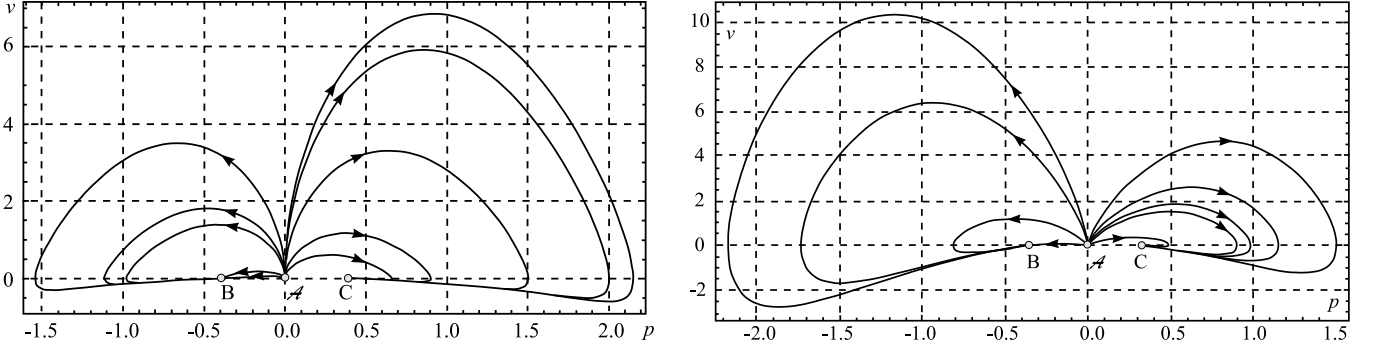


FIG. 1: The phase portrait for the case $n = 2$ (left panel) and $n = 4/3$ (right panel), $\alpha = 1$. \mathcal{A} is a repulsive node, B, C denote the asymptotic points (12) at $z \rightarrow \mp\infty$, respectively. The different curves correspond to different initial values of the variable y .

The behavior of the solutions in the neighborhood of the fixed point (and the curves of energy) can be estimated as follows: from the second and third equations of the system (8), it is possible to obtain an equation near the fixed point

$$\frac{dv}{dp} = \frac{1}{2} \frac{v}{p} \Rightarrow v = D\sqrt{p},$$

where D is an integration constant. This is a set of curves escaping from the node $p = 0, v = 0$. Specifying the boundary conditions in neighborhood of the fixed point \mathcal{A} , one can then draw a phase portrait of the system (8), presented in Fig. 1. As a further example, the model with $n = 4/3$ is also presented in this figure.

The asymptotical form of the solution for arbitrary n is given by

$$y_\infty = k_n |z|, \quad k_n = \left[\frac{12}{\alpha(1 - \frac{2n}{5})20^n} \right]^{\frac{1}{2(n-1)}}. \quad (12)$$

One can see that there is an upper bound for the parameter n , $n < 5/2$. Taking into account the condition from (11) one can conclude that such a type of thick brane solution can exist only in the range $0 < n < 5/2, n \neq 1$.

Let us now discuss the corresponding distribution of the effective energy density $\hat{T}_0^0 = -3(y'' + 2y'^2)$ for the case $n = 2$, shown in Fig. 2. Asymptotically \hat{T}_0^0 goes to a constant negative value: $\hat{T}_{0(\pm\infty)}^0 \rightarrow -6y_\infty'^2 = -0.9$, and the 5-dimensional scalar curvature $R^{(5)} = 8y'' + 20y'^2$ goes to a positive constant, $R_{(\pm\infty)}^{(5)} = 3$. Thus this corresponds to an asymptotical anti-de Sitter solution.

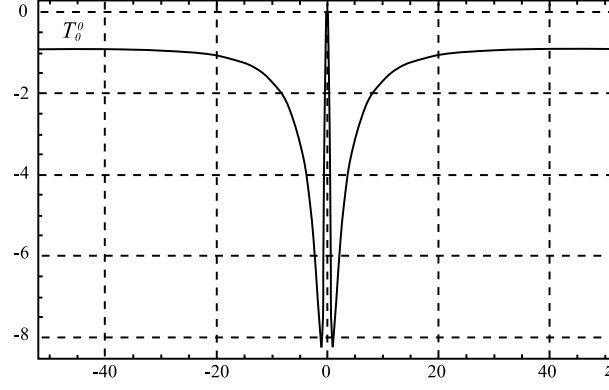


FIG. 2: The effective energy density \hat{T}_0^0 for the case $n = 2$, $\alpha = 1$. The point $\hat{T}_0^0 = 0$ has been placed at $z = 0$ by shifting $z \rightarrow z - z_0$.

III. TRAPPING OF MATTER

In this section we consider the trapping of a test scalar field on the brane considered above. For this purpose, let us use the approach suggested in the paper [20]. We consider the test complex scalar field χ with the Lagrangian

$$L_\chi = \frac{1}{2} \partial_A \chi^* \partial^A \chi - \frac{1}{2} m_0^2 \chi^* \chi,$$

where m_0 is the mass of the test field. Using this Lagrangian, we find the equation for the scalar field

$$\frac{1}{\sqrt{-5}g} \frac{\partial}{\partial x^A} \left(\sqrt{-5}g g^{AB} \frac{\partial \chi}{\partial x^B} \right) = -m_0^2 \chi. \quad (13)$$

Here χ is a function of all coordinates, $\chi = \chi(x^A)$. Taking into account that the canonically conjugate momenta $p_\mu = (E, \vec{p})$ are integrals of motion, we will seek a solution in the form

$$\chi(x^A) = X(z) \exp(-ip_\mu x^\mu).$$

Inserting this ansatz into Eq. (13), leads to the equation for $X(z)$

$$X'' + \sqrt{-5}g(p^\mu p_\mu - m_0^2)X = 0,$$

or, taking into account that $p^\mu p_\mu = e^{-2y} (E^2 - \vec{p}^2)$, we obtain

$$X'' + [(E^2 - \vec{p}^2) e^{2y} - m_0^2 e^{4y}] X = 0.$$

According to Eq. (12), asymptotically $y_\infty = k_n |z|$, $k_n > 0$. That is why the dominant term in the above equation will be the term with e^{4y} . Thus

$$X'' - m_0^2 e^{4k_n |z|} X = 0$$

with the asymptotically decaying solution

$$X_\infty \approx C \sqrt{\frac{2k_n}{m_0}} e^{-2k_n |z|} \exp\left(-\frac{m_0}{2k_n} e^{2k_n |z|}\right), \quad (14)$$

where C is an integration constant.

As a necessary condition for the trapping of matter on the brane, one should require finiteness of the field energy per unit 3-volume of the brane, i.e.,

$$E_{\text{tot}}[\chi] = \int_{-\infty}^{\infty} T_0^0 \sqrt{-5}g dz = \int_{-\infty}^{\infty} e^{4k_n |z|} \left[e^{-2k_n |z|} (E^2 + \vec{p}^2) X^2 + m_0^2 X^2 + X'^2 \right] dz < \infty, \quad (15)$$

and also finiteness of the norm of the field χ

$$||\chi||^2 = \int_{-\infty}^{\infty} \sqrt{-^5g} \chi^* \chi dz = \int_{-\infty}^{\infty} e^{4k_n|z|} X^2 dz.$$

From the solution (14) it is evident that both E_{tot} and $||\chi||$ converge asymptotically.

IV. CONCLUSION

We have considered the 5-dimensional thick brane model in $f(R) \sim R^n$ theory. It was shown that regular solutions with asymptotic (12) exist in the range $0 < n < 5/2$, $n \neq 1$. Space is asymptotically anti-de Sitter in such a brane model.

A special feature of the model is the presence of the fixed point \mathcal{A} . Since Eq. (7) is invariant under the shift of the independent variable $z \rightarrow z + z_0$, the position of the brane is arbitrary and it can be placed at any point on the axis z , including $z = 0$. The most natural assumption seems to be that the brane is situated at the fixed point \mathcal{A} at $z = 0$. Then $y(0) = \text{const}$, $y'(0) = 0$, $y''(0) = 0$ holds on the brane.

The presence of the repulsive fixed point \mathcal{A} (node) in the system allows to set the boundary conditions arbitrarily in the neighborhood of the fixed point. Anyway, all solutions will escape from \mathcal{A} and tend to the asymptotic value (12). It allows to not provide any special conditions for the model parameters (fine-tuning conditions) that is typically necessary for other brane models (see, for example, Ref. [20]).

Consideration of the behavior of a test scalar field in the bulk has shown that such a field is trapped by the $f(R)$ -brane. Note that the trapping is purely gravitational.

For $n = 2$, this $f(R)$ theory has also been employed for the thin brane model [18]. In the vicinity of the brane (located at $z = 0$) a similar behaviour of the metric function is found, $y(z) \sim z^3$. However, at some finite value z_s , a singularity is encountered. Thus the thin brane model appears to be plagued by bulk singularities, very much in contrast to the thick brane model considered in this paper.

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